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Level-augmented uniform designs

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Abstract
Most of existing augmented designs are to add some runs in the follow-up stages. While in many cases, the level of factors should be augmented and these augmented designs are called level-augmented designs. According to whether the experimental domain is extended or not, they can be divided into range-extended and range-fixed level-augmented designs. For different types of initial designs, the symmetrical and asymmetrical level-augmented designs are discussed, respectively. Based on the property of robustness, a uniformity criterion is a suitable choice to obtain an optimal level-augmented design when the model is unknown. In this paper, the wrap-around L_2-discrepancy (WD) is chosen as the uniformity measure. We give the expressions and the tight lower bounds of WD of level-augmented designs under some special parameters. A method to construct a special case of symmetrical level-augmented designs is given. Some examples and level-augmented uniform designs are also provided.

Keywords Level-augmented design · Lower bound · Wrap-around L_2-discrepancy

1 Introduction

The follow-up strategy is popularly used in practical applications. It adds a fraction to the initial design to obtain more information. At the initial stage of an experiment, the design is chosen as an optimal or nearly optimal under some design criterion. After analyzing the data of the initial design, the existing data may not be enough to achieve the intended purpose and hence the follow-up design is needed. For example, Wang et al. (2010) developed an anvil pre-formed gasket system, and it was necessary to extend the range of the factor cell pressure-press from the original experimental

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upper bound 6.5 MN into 8.0 MN in the follow-up stage, for increasing the response the maximum pressure in the conventional anvil-gasket system. Dilipkumar et al. (2011) studied the effects of different nutrient elements on the inulinase production. The initial experiment showed that in the follow-up stage the experimental levels of some important factors should be extended in the initial experimental range for obtaining higher productivity. It is known that the experimental domain in Wang et al. (2010) was extended as well as the experimental levels, and the experimental domain in Dilipkumar et al. (2011) was unchanged while the experimental levels were extended. We call a design as the level-augmented design, when the numbers of levels of some factors are augmented. Furthermore, if the experimental domain is extended in the follow-up stage, the level-augmented design can be called the range-extended level-augmented design (RELAD), otherwise, the range-fixed level-augmented design (RFLAD). In this paper, we will consider both the types of RELAD and RFLAD.

In the initial stage and the follow-up stage of many experiments, they often have no prior information for the relationships between the factors and the response. In those cases, the uniform designs, proposed by Fang and T. (1980) and Wang and Fang (1981), are a suitable choice for arranging the experiments. Because of the robustness and flexibility, uniform designs have been widely applied in manufacturing, system engineering, pharmaceutics and natural sciences. Its main idea is to scatter design points uniformly on the experimental domain. A commonly used measure of the uniformity of a design is the discrepancy. The wrap-around L_2-discrepancy (WD, Hickernell 1998) has been widely used in the literature. Based on the WD, there were some research of follow-up designs, such as Qin et al. (2013), Qin et al. (2016), Gou et al. (2018), Yang et al. (2017) and Yang et al. (2019). Qin et al. (2016) used the WD to measure the uniformity of two-level augmented designs, which added some runs for the initial two-level designs. Gou et al. (2018) used the WD to measure the uniformity of mixed two- and three-level augmented designs. Yang et al. (2019) augmented the number of runs and factors for mixed two- and three-level designs under WD. However, in those literatures, the number of levels of factors of the added portion is the same as that of the initial design. In those cases of RELADs and RFLADs, the numbers of levels of some factors are augmented in the follow-up stages. Under a uniformity criterion, the added points for each case should be scattered uniformly in the whole experimental domain coupled with the initial points. In this paper, we discuss the level-augmented designs under WD including both RELADs and RFLADs.

The rest of the paper is organized as follows. In Sect. 2, the definitions of range-extended and range-fixed level-augmented designs are given and the corresponding expressions of WD are derived. Section 3 gives the lower bounds of WD for level-augmented designs under some special parameters. Section 4 presents a method to construct one kind of level-augmented designs where the parameters satisfy some conditions. We also show some examples in this section. Some conclusions and discussions are summarized in Sect. 5. All the proofs of the theorems are given in the Appendix. In the supplementary materials, we show the proofs of all the propositions, some additional results and some uniform and nearly uniform level-augmented designs.

2 Level-augmented designs

Some notations are given first. A design $D(n; q_1, \ldots, q_m)$ is an $n \times m$ matrix $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$, each column $\boldsymbol{x}_i$ takes values from $\{1, \ldots, q_i\}, i = 1, \ldots, m$. If some q_i's are equal, we denote it as an asymmetrical design $D(n; q_1^{r_1}, \ldots, q_s^{r_s})$, where $m = \sum_{i=1}^{s} r_i$. If all the q_i's are equal, we call this design as a symmetrical design and denote it as $D(n; q^m)$. Denote all of the $D(n; q^m)$ and $D(n; q_1^{r_1}, \ldots, q_s^{r_s})$ by $\mathcal{D}(n; q^m)$ and $\mathcal{D}(n; q_1^{r_1}, \ldots, q_s^{r_s})$, respectively. If each level in each column of $D(n; q_1, \ldots, q_m)$ occurs equally often, we call it U-type design and denote it as $U(n; q_1, \ldots, q_m)$. The U-type design $U(n; q^m)$ and $U(n; q_1^{r_1}, \ldots, q_s^{r_s})$ can be defined similarly, as well as the $\mathcal{U}(n; q^m)$ and $\mathcal{U}(n; q_1^{r_1}, \ldots, q_s^{r_s})$. For each design $\boldsymbol{d} \in \mathcal{U}(n; q_1^{r_1}, \ldots, q_s^{r_s})$, the n runs of $\boldsymbol{d}$ can be transformed into the n points on $C^m = [0, 1]^m$ by mapping $f : x_{ik} \rightarrow (2x_{ik} - 1)/(2q_k), i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m$. The squared WD-value of $\boldsymbol{d} \in \mathcal{U}(n; q_1^{r_1}, \ldots, q_s^{r_s})$ is

$$WD^2(\boldsymbol{d}) = -\left(\frac{4}{3}\right)^m + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{k=1}^{m} \left(\frac{3}{2} - |u_{ik} - u_{jk}|(1 - |u_{ik} - u_{jk}|)\right), \quad (1)$$

where $u_{ik} = (2x_{ik} - 1)/(2q_k), i = 1, 2, \ldots, n, k = 1, 2, \ldots, m$.

Given an initial design $\boldsymbol{d}_0$, the follow-up stage may not only add additional runs but also augment the number of levels of some factors.

It will be shown that the number of levels of the m factors in $\boldsymbol{d}_0$ affects the property of the augmented design. In practical applications, for RELADs, the augmentation of the number of levels is often augmented by one, such as Wang et al. (2010). For RFLADs, we fix the experimental range, the augmentation of the number of levels may be larger than one, such as Dilipkumar et al. (2011). Let n_1 be the number of the added runs, m_1 and m_2 be the number of factors which need not to and need to augment the number of levels, respectively. Then, we focus on those cases and give the definitions of the two types of level-augmented designs.

Definition 1 (1) An augmented design $D_1 = (\boldsymbol{d}_0^T\ \boldsymbol{d}_1^T)^T \in \mathcal{U}(n+n_1; (q+1)^m)$ is called the symmetrical RELAD, if the initial design $\boldsymbol{d}_0 \in \mathcal{U}(n; q^{m_2}(q+1)^{m_1})$, the follow-up stage $\boldsymbol{d}_1 \in \mathcal{D}(n_1; (q+1)^m)$ and $m = m_1 + m_2$. Let $\mathcal{L}_e(n + n_1; (q+1)^m)$ denote all the symmetrical RELADs. The design $D_2 = (\boldsymbol{d}_0^T\ \boldsymbol{d}_1^T)^T \in \mathcal{U}(n + n_1; q^{m_1}(q+1)^{m_2})$ is called the asymmetrical RELAD, if $\boldsymbol{d}_0 \in \mathcal{U}(n; q^m)$ and $\boldsymbol{d}_1 \in \mathcal{D}(n_1; q^{m_1}(q+1)^{m_2})$. Let $\mathcal{L}_e(n + n_1; q^{m_1}(q+1)^{m_2})$ denote all the asymmetrical RELADs.

(2) An augmented design $D_1' = (\boldsymbol{d}_0^T\ \boldsymbol{d}_1^T)^T \in \mathcal{U}(n + n_1; (2+q)^m)$ is called the symmetrical RFLAD, if the initial design $\boldsymbol{d}_0 \in \mathcal{U}(n; 2^{m_2}(2+q)^{m_1})$, in which the levels $\{1, 2\}$ become $\{1, 2+q\}$ for the m_2 level-augmented factors, the follow-up stage $\boldsymbol{d}_1 \in \mathcal{D}(n_1; (2+q)^m)$ and $m = m_1 + m_2$. Let $\mathcal{L}_f(n + n_1; (2+q)^m)$ denote all the symmetrical RFLADs. The design $D_2' = (\boldsymbol{d}_0^T\ \boldsymbol{d}_1^T)^T \in \mathcal{U}(n+n_1; 2^{m_1}(2+q)^{m_2})$ is called the asymmetrical RFLAD, if $\boldsymbol{d}_0 \in \mathcal{U}(n; 2^m)$, in which the levels $\{1, 2\}$ become $\{1, 2+q\}$ for the m_2 level-augmented factors, and $\boldsymbol{d}_1 \in \mathcal{D}(n_1; 2^{m_1}(2+q)^{m_2})$. Let $\mathcal{L}_f(n + n_1; 2^{m_1}(2+q)^{m_2})$ denote all the asymmetrical RFLADs.

To understand Definition 1, we give an example as follows.

Example 1 Suppose the initial design $\boldsymbol{d}_0 \in \mathcal{U}(6; 2^1 3^3)$ is

$$\boldsymbol{d}_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 3 \\ 2 & 1 & 2 & 3 \\ 2 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \end{pmatrix}.$$

We add an additional portion $\boldsymbol{d}_1 \in \mathcal{D}(3; 3^4)$ as follows,

$$\boldsymbol{d}_1 = \begin{pmatrix} 3 & 1 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 3 & 3 & 2 & 1 \end{pmatrix}.$$

In this case, $n = 6, n_1 = 3, m_1 = 3, m_2 = 1, q = 2$. We augment the number of levels of the m_2 factors from two levels to three levels by adding n_1 runs. Then we expand the experimental range and $D_1 = (\boldsymbol{d}_0^T \ \ \boldsymbol{d}_1^T)^T$ is the symmetrical RELAD. Moreover, given $\boldsymbol{d}_0$, we transform $\boldsymbol{d}_0$ into $\boldsymbol{d}'_0$ by Definition 1(2) and add an additional portion $\boldsymbol{d}'_1$, where

$$\boldsymbol{d}'_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 3 \\ 3 & 1 & 2 & 3 \\ 3 & 2 & 3 & 1 \\ 3 & 3 & 1 & 2 \end{pmatrix}, \quad \boldsymbol{d}'_1 = \begin{pmatrix} 2 & 1 & 3 & 2 \\ 2 & 2 & 1 & 3 \\ 2 & 3 & 2 & 1 \end{pmatrix}.$$

In this case, $n = 6$, $n_1 = 3$, $m_1 = 3$, $m_2 = 1$, $q = 1$. We fix the experimental range and add three points in the domain. Then $D'_1 = (\boldsymbol{d}'^T_0 \ \ \boldsymbol{d}'^T_1)^T$ is the symmetrical RFLAD.

From Definition 1, for both range-extended and range-fixed cases, the symmetrical level-augmented design augments the initial mixed-level design to the symmetrical design through augmenting the number of levels, and the asymmetrical level-augmented design augments the symmetrical initial design to the asymmetrical design. Generally, a level-augmented design can augment an initial design in $\mathcal{U}(n; q_1^{r_1}, \cdots, q_s^{r_s})$ to a design in $\mathcal{U}(n + n_1; (q_1 + t_1)^{r_1}, \cdots, (q_s + t_s)^{r_s})$ with $t_i \geq 0$. We only consider the special cases in Definition 1 since they are common in practice. The RELADs augment the number of levels of the m_2 factors from q to $q + 1$ levels, which means that each of the experimental range of those m_2 factors is extended. The RFLADs augment the number of levels of the m_2 factors from 2 to $2+q$ levels. However, the levels $\{1, 2\}$ turn into $\{1, 2 + q\}$ for the initial portion and the added portion takes values from $\{1, 2, \ldots, 2 + q\}$. Hence the experimental range is not changed for these factors. Usually, m_2 may be 1 or 2.

Note that for all the cases, the initial designs and the resulting augmented designs are required to be U-type designs owing to its good property. Then, the number of runs

n and n_1 in the initial design and the follow-up stage may have some requirements. For example, for the $(q+1)$-level symmetrical RELAD D_1, let the initial number of runs $n = k_1 \cdot q(q+1)$, where k_1 is a positive integer, and the number of the added runs $n_1 = n_{11} + n_{12}$, where $n_{11} = n/q$, $n_{12} = k_2 \cdot (q+1)$ and k_2 is a nonnegative integer. For the asymmetrical RELAD D_2, let the initial number of runs $n = k_3 \cdot q$, where k_3 is a positive integer, and the number of the additional runs $n_1 = n_{11} + n_{12}$, where $n_{11} = n/q$, $n_{12} = k_4 \cdot (q+1)$ and k_4 is a nonnegative integer such that n_1/q is also a nonnegative integer. For the $(2+q)$-level symmetrical RFLAD D_1', let the initial number of runs $n = l_1 \cdot 2(2+q)$, where l_1 is a positive integer, and the number of the added runs $n_1 = n_{11} + n_{12}$, where $n_{11} = q \cdot (n/2)$, $n_{12} = l_2 \cdot (2+q)$ and l_2 is a nonnegative integer. For the asymmetrical RFLAD D_2', let the initial number of runs $n = l_3 \cdot 2$, where l_3 is a positive integer, and the number of the additional runs $n_1 = n_{11} + n_{12}$, where $n_{11} = q \cdot (n/2)$, $n_{12} = l_4 \cdot (2+q)$ and l_4 is a nonnegative integer such that $n_1/[2(2+q)]$ is also a nonnegative integer. Moreover, it should be mentioned that the restriction of level-augmented designs to be U-type designs can be relaxed, i.e., one can augment any number of runs n_1 based on the initial design. In the rest of the paper, we will consider the cases when level-augmented designs are U-type designs.

Definition 2 (1) A level-augmented design from $\mathcal{L}_e(n+n_1; (q+1)^m)$ or $\mathcal{L}_e(n+n_1; q^{m_1}(q+1)^{m_2})$ is called the range-extended level-augmented uniform design, if it has the smallest WD-value among $\mathcal{L}_e(n+n_1; (q+1)^m)$ or $\mathcal{L}_e(n+n_1; q^{m_1}(q+1)^{m_2})$.

(2) A level-augmented design from $\mathcal{L}_f(n+n_1; (2+q)^m)$ or $\mathcal{L}_f(n+n_1; 2^{m_1}(2+q)^{m_2})$ is called the range-fixed level-augmented uniform design, if it has the smallest WD-value among $\mathcal{L}_f(n+n_1; (2+q)^m)$ or $\mathcal{L}_f(n+n_1; 2^{m_1}(2+q)^{m_2})$.

From the analytical expression of the squared WD-value in (1), it is easy to see that the WD-value is only a function of the products of $\alpha_{ij}^k \equiv |u_{ik} - u_{jk}|(1 - |u_{ik} - u_{jk}|)$, $i, j = 1, \ldots, n$, $i \neq j$ and $k = 1, \ldots, m$. For any i and j, denote the distribution of $\{\alpha_{ij}^k,\ k = 1, \ldots, m\}$ by F_{ij}^{α}. According to Fang et al. [8], when q is even, the α_{ij}^k-values can only take $q/2 + 1$ possible values, i.e. $0, 2(2q-2)/(4q^2), 4(2q-4)/(4q^2), \ldots, q^2/(4q^2)$; when q is odd, they can only take $(q+1)/2$ possible values, i.e. $0, 2(2q-2)/(4q^2), 4(2q-4)/(4q^2), \ldots, (q-1)(q+1)/(4q^2)$. It implies that the expressions of WD for the level-augmented designs shall vary with the parity of q. For both range-extended and range-fixed cases, we obtain the expressions of the squared WD-value of the symmetrical and asymmetrical level-augmented designs as follows.

2.1 Range-extended level-augmented designs

We first give the expression of the squared WD-value of a symmetrical RELAD in Proposition 1.

Proposition 1 *For any symmetrical RELAD $D_1 \in \mathcal{L}_e(n+n_1;(q+1)^m)$, if q is even, we have*

$$
\begin{aligned}
WD^2(D_1) = &-\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m \\
&+ \frac{1}{(n+n_1)^2}\Bigg[\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\prod_{l=0}^{q/2}\left(\frac{3}{2}-\frac{2l(2(q+1)-2l)}{4(q+1)^2}\right)^{\varphi_{ijl}} \times \\
&\prod_{r=0}^{q/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\lambda_{ijr}} \\
&+ \sum_{i=n+1}^{n+n_1}\sum_{j(\neq i)=n+1}^{n+n_1}\prod_{l=0}^{q/2}\left(\frac{3}{2}-\frac{2l(2(q+1)-2l)}{4(q+1)^2}\right)^{\varphi'_{ijl}} \times \\
&\prod_{r=0}^{q/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\lambda'_{ijr}} \\
&+ 2\sum_{i=1}^{n}\sum_{j=n+1}^{n+n_1}\prod_{l=0}^{q/2}\left(\frac{3}{2}-\frac{2l(2(q+1)-2l)}{4(q+1)^2}\right)^{\nu_{ijl}} \times \\
&\prod_{r=0}^{q/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\tau_{ijr}}\Bigg];
\end{aligned}
$$

if q is odd, we have

$$
\begin{aligned}
WD^2(D_1) = &-\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m \\
&+ \frac{1}{(n+n_1)^2}\Bigg[\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\prod_{l=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2l(2(q+1)-2l)}{4(q+1)^2}\right)^{\varphi_{ijl}} \times \\
&\prod_{r=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\lambda_{ijr}} \\
&+ \sum_{i=n+1}^{n+n_1}\sum_{j(\neq i)=n+1}^{n+n_1}\prod_{l=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2l(2(q+1)-2l)}{4(q+1)^2}\right)^{\varphi'_{ijl}} \times \\
&\prod_{r=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\lambda'_{ijr}}
\end{aligned}
$$

$$+2\sum_{i=1}^{n}\sum_{j=n+1}^{n+n_1}\prod_{l=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2l(2(q+1)-2l)}{4(q+1)^2}\right)^{\nu_{ijl}}\times$$
$$\prod_{r=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\tau_{ijr}}\Bigg]. \quad (2)$$

Define $c = q/2$ *and* $(q+1)/2$ *for even* q *and odd* q, *respectively,* $\sum_{l=0}^{c}\varphi_{ijl} = m_1$, $\sum_{r=0}^{c}\lambda_{ijr} = m_2$, $\varphi_{ij0} = \#\{k : u_{ik} = u_{jk}, k = 1, 2, \ldots, m_1\}$, $\varphi_{ijl} = \#\{k : \alpha_{ij}^k = \frac{2l(2(q+1)-2l)}{4(q+1)^2}, k = 1, 2, \ldots, m_1\}, l = 1, 2, \ldots, c$, $\lambda_{ij0} = \#\{k : u_{ik} = u_{jk}, k = m_1 + 1, \ldots, m\}$, $\lambda_{ijr} = \#\{k : \alpha_{ij}^k = \frac{2r(2(q+1)-2r)}{4(q+1)^2}, k = m_1+1, \ldots, m\}, r = 1, 2, \ldots, c$ *for* $i, j(\neq i) = 1, 2, \ldots, n$; $\sum_{l=0}^{c}\varphi'_{ijl} = m_1$, $\sum_{r=0}^{c}\lambda'_{ijr} = m_2$, $\varphi'_{ij0} = \#\{k : u_{ik} = u_{jk}, k = 1, 2, \ldots, m_1\}$, $\varphi'_{ijl} = \#\{k : \alpha_{ij}^k = \frac{2l(2(q+1)-2l)}{4(q+1)^2}, k = 1, 2, \ldots, m_1\}, l = 1, 2, \ldots, c$, $\lambda'_{ij0} = \#\{k : u_{ik} = u_{jk}, k = m_1 + 1, \ldots, m\}$, $\lambda'_{ijr} = \#\{k : \alpha_{ij}^k = \frac{2r(2(q+1)-2r)}{4(q+1)^2}, k = m_1 + 1, \ldots, m\}, r = 1, 2, \cdots, c$ *for* $i, j(\neq i) = n+1, \ldots, n+n_1$; $\sum_{l=0}^{c}\nu_{ijl} = m_1$, $\sum_{r=0}^{c}\tau_{ijr} = m_2$, $\nu_{ij0} = \#\{k : u_{ik} = u_{jk}, k = 1, 2, \cdots, m_1\}$, $\nu_{ijl} = \#\{k : \alpha_{ij}^k = \frac{2l(2(q+1)-2l)}{4(q+1)^2}, k = 1, 2, \ldots, m_1\}, l = 1, 2, \ldots, c$, $\tau_{ij0} = \#\{k : u_{ik} = u_{jk}, k = m_1 + 1, \ldots, m\}$, $\tau_{ijr} = \#\{k : \alpha_{ij}^k = \frac{2r(2(q+1)-2r)}{4(q+1)^2}, k = m_1+1, \ldots, m\}, r = 1, 2, \ldots, c$ *for* $i = 1, 2, \ldots, n$, $j = n+1, \ldots, n+n_1$, *and* $\#\{S\}$ *is the number of elements in the set* S.

The proof of Proposition 1 is given in the Supplementary Material A1. Hereafter, we will use the parameters defined in Proposition 1. In practical applications, mixed two- and three-level designs and mixed three- and four-level designs are commonly used for the initial designs, so the corresponding symmetrical RELADs are valuable. For three-level RELAD, the expression of the squared WD-value is given in Corollary 1. In addition, the corresponding expression of WD for four-level RELAD is given in the Supplementary Material B1.

Corollary 1 *For any initial design* $\boldsymbol{d}_0 \in \mathcal{U}(n; 2^{m_2}3^{m_1})$ *and the corresponding symmetrical RELAD* $D_1 \in \mathcal{L}_e(n + n_1; 3^m)$, *we have*

$$WD^2(D_1) = -\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m$$
$$+\frac{1}{(n+n_1)^2}\Bigg[\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(\frac{3}{2}\right)^{\varphi_{ij0}+\lambda_{ij0}}\left(\frac{23}{18}\right)^{m_1+m_2-\varphi_{ij0}-\lambda_{ij0}}$$
$$+\sum_{i=n+1}^{n+n_1}\sum_{j(\neq i)=n+1}^{n+n_1}\left(\frac{3}{2}\right)^{\varphi'_{ij0}+\lambda'_{ij0}}\left(\frac{23}{18}\right)^{m_1+m_2-\varphi'_{ij0}-\lambda'_{ij0}}$$

$$+2\sum_{i=1}^{n}\sum_{j=n+1}^{n+n_1}\left(\frac{3}{2}\right)^{\nu_{ij0}+\tau_{ij0}}\left(\frac{23}{18}\right)^{m_1+m_2-\nu_{ij0}-\tau_{ij0}}\Bigg]. \tag{3}$$

Similarly, we can also derive the expression of the squared WD-value for the asymmetrical RELAD as follows.

Proposition 2 *For any asymmetrical RELAD $D_2 \in \mathcal{L}_e(n+n_1; q^{m_1}(q+1)^{m_2})$, if q is even, we have*

$$\begin{aligned} WD^2(D_2) = &-\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m \\ &+\frac{1}{(n+n_1)^2}\Bigg[\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\prod_{l=0}^{q/2}\left(\frac{3}{2}-\frac{2l(2q-2l)}{4q^2}\right)^{\varphi_{ijl}}\times \\ &\prod_{r=0}^{q/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\lambda_{ijr}} \\ &+\sum_{i=n+1}^{n+n_1}\sum_{j(\neq i)=n+1}^{n+n_1}\prod_{l=0}^{q/2}\left(\frac{3}{2}-\frac{2l(2q-2l)}{4q^2}\right)^{\varphi'_{ijl}}\times \\ &\prod_{r=0}^{q/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\lambda'_{ijr}} \\ &+2\sum_{i=1}^{n}\sum_{j=n+1}^{n+n_1}\prod_{l=0}^{q/2}\left(\frac{3}{2}-\frac{2l(2q-2l)}{4q^2}\right)^{\nu_{ijl}}\times \\ &\prod_{r=0}^{q/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\tau_{ijr}}\Bigg]; \end{aligned}$$

if q is odd, we have

$$\begin{aligned} WD^2(D_2) = &-\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m \\ &+\frac{1}{(n+n_1)^2}\Bigg[\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\prod_{l=0}^{(q-1)/2}\left(\frac{3}{2}-\frac{2l(2q-2l)}{4q^2}\right)^{\varphi_{ijl}}\times \\ &\prod_{r=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\lambda_{ijr}} \\ &+\sum_{i=n+1}^{n+n_1}\sum_{j(\neq i)=n+1}^{n+n_1}\prod_{l=0}^{(q-1)/2}\left(\frac{3}{2}-\frac{2l(2q-2l)}{4q^2}\right)^{\varphi'_{ijl}}\times \end{aligned}$$

$$\prod_{r=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\lambda'_{ijr}}$$
$$+2\sum_{i=1}^{n}\sum_{j=n+1}^{n+n_1}\prod_{l=0}^{(q-1)/2}\left(\frac{3}{2}-\frac{2l(2q-2l)}{4q^2}\right)^{\nu_{ijl}}\times$$
$$\prod_{r=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+1)-2r)}{4(q+1)^2}\right)^{\tau_{ijr}}\Bigg]. \tag{4}$$

The proof of Proposition 2 is given in the Supplementary Material A2. In addition, two-level and three-level initial designs are also widely used in practice, hence we give the expression of the squared WD-value for mixed two- and three-level RELAD in Corollary 2 and the corresponding expression of WD for mixed three- and four-level RELAD is shown in the Supplementary Material B2.

Corollary 2 *For any initial design $\boldsymbol{d}_0 \in \mathcal{U}(n; 2^m)$ and the corresponding asymmetrical RELAD $D_2 \in \mathcal{L}_e(n+n_1; 2^{m_1}3^{m_2})$, we have*

$$\begin{aligned}WD^2(D_2) =& -\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m\\
&+\frac{1}{(n+n_1)^2}\Bigg[\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(\frac{3}{2}\right)^{\varphi_{ij0}+\lambda_{ij0}}\left(\frac{5}{4}\right)^{m_1-\varphi_{ij0}}\left(\frac{23}{18}\right)^{m_2-\lambda_{ij0}}\\
&+\sum_{i=n+1}^{n+n_1}\sum_{j(\neq i)=n+1}^{n+n_1}\left(\frac{3}{2}\right)^{\varphi'_{ij0}+\lambda'_{ij0}}\left(\frac{5}{4}\right)^{m_1-\varphi'_{ij0}}\left(\frac{23}{18}\right)^{m_2-\lambda'_{ij0}}\\
&+2\sum_{i=1}^{n}\sum_{j=n+1}^{n+n_1}\left(\frac{3}{2}\right)^{\nu_{ij0}+\tau_{ij0}}\left(\frac{5}{4}\right)^{m_1-\nu_{ij0}}\left(\frac{23}{18}\right)^{m_2-\tau_{ij0}}\Bigg].\end{aligned} \tag{5}$$

2.2 Range-fixed level-augmented designs

For the symmetrical RFLAD, we give the expression of the squared WD-value as follows.

Proposition 3 *For any symmetrical RFLAD $D'_1 \in \mathcal{L}_f(n+n_1; (2+q)^m)$, if q is even, we have*

$$\begin{aligned}WD^2(D'_1) =& -\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m\\
&+\frac{1}{(n+n_1)^2}\Bigg[\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(\frac{3}{2}\right)^{\lambda_{ij0}}\left(\frac{3}{2}-\frac{q+1}{(q+2)^2}\right)^{\lambda_{ij1}}\times\\
&\prod_{l=0}^{q/2+1}\left(\frac{3}{2}-\frac{2l(2(q+2)-2l)}{4(q+2)^2}\right)^{\varphi_{ijl}}\end{aligned}$$

$$+\sum_{i=n+1}^{n+n_1}\sum_{j(\neq i)=n+1}^{n+n_1}\prod_{l=0}^{q/2+1}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\varphi'_{ijl}}\times$$
$$\prod_{r=0}^{q/2+1}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\lambda'_{ijr}}$$
$$+2\sum_{i=1}^{n}\sum_{j=n+1}^{n+n_1}\prod_{l=0}^{q/2+1}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\nu_{ijl}}\times$$
$$\prod_{r=0}^{q/2+1}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\tau_{ijr}}\Bigg];$$

if q is odd, we have

$$WD^2(D'_1)=-\left(\frac{4}{3}\right)^m+\frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m$$
$$+\frac{1}{(n+n_1)^2}\Bigg[\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(\frac{3}{2}\right)^{\lambda_{ij0}}\left(\frac{3}{2}-\frac{q+1}{(q+2)^2}\right)^{\lambda_{ij1}}\times$$
$$\prod_{l=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2l(2(q+2)-2l)}{4(q+2)^2}\right)^{\varphi_{ijl}}$$
$$+\sum_{i=n+1}^{n+n_1}\sum_{j(\neq i)=n+1}^{n+n_1}\prod_{l=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\varphi'_{ijl}}\times$$
$$\prod_{r=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\lambda'_{ijr}}$$
$$+2\sum_{i=1}^{n}\sum_{j=n+1}^{n+n_1}\prod_{l=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\nu_{ijl}}\times$$
$$\prod_{r=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\tau_{ijr}}\Bigg]. \tag{6}$$

The proof of Proposition 3 is given in the Supplementary Material A3. If the initial design is a mixed two- and three-level design, the expression of the squared WD-value of the symmetrical three-level RFLAD is the same as (3). In addition, if the initial design is a mixed two- and four-level design, which is also commonly used in practice, the corresponding expression of WD for four-level RFLAD is shown in the Supplementary Material B3.

Similarly, we can obtain the expression of the squared WD-value for the asymmetrical RFLAD as follows.

Proposition 4 *For any asymmetrical RFLAD $D_2' \in \mathcal{L}_f(n+n_1; 2^{m_1}(2+q)^{m_2})$, if q is even, we have*

$$\begin{aligned} WD^2(D_2') &= -\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m \\ &+ \frac{1}{(n+n_1)^2}\left[\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(\frac{3}{2}\right)^{\varphi_{ij0}}\left(\frac{5}{4}\right)^{\varphi_{ij1}}\left(\frac{3}{2}\right)^{\lambda_{ij0}}\left(\frac{3}{2}-\frac{(q+1)}{(q+2)^2}\right)^{\lambda_{ij1}}\right. \\ &+ \sum_{i=n+1}^{n+n_1}\sum_{j(\neq i)=n+1}^{n+n_1}\left(\frac{3}{2}\right)^{\varphi'_{ij0}}\left(\frac{5}{4}\right)^{\varphi'_{ij1}}\prod_{r=0}^{(q+2)/2}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\lambda'_{ijr}} \\ &\left. + 2\sum_{i=1}^{n}\sum_{j=n+1}^{n+n_1}\left(\frac{3}{2}\right)^{\nu_{ij0}}\left(\frac{5}{4}\right)^{\nu_{ij1}}\prod_{r=0}^{(q+2)/2}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\tau_{ijr}}\right]; \end{aligned}$$

if q is odd, we have

$$\begin{aligned} WD^2(D_2') &= -\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m \\ &+ \frac{1}{(n+n_1)^2}\left[\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(\frac{3}{2}\right)^{\varphi_{ij0}}\left(\frac{5}{4}\right)^{\varphi_{ij1}}\left(\frac{3}{2}\right)^{\lambda_{ij0}}\left(\frac{3}{2}-\frac{(q+1)}{(q+2)^2}\right)^{\lambda_{ij1}}\right. \\ &+ \sum_{i=n+1}^{n+n_1}\sum_{j(\neq i)=n+1}^{n+n_1}\left(\frac{3}{2}\right)^{\varphi'_{ij0}}\left(\frac{5}{4}\right)^{\varphi'_{ij1}}\prod_{r=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\lambda'_{ijr}} \\ &\left. + 2\sum_{i=1}^{n}\sum_{j=n+1}^{n+n_1}\left(\frac{3}{2}\right)^{\nu_{ij0}}\left(\frac{5}{4}\right)^{\nu_{ij1}}\prod_{r=0}^{(q+1)/2}\left(\frac{3}{2}-\frac{2r(2(q+2)-2r)}{4(q+2)^2}\right)^{\tau_{ijr}}\right]. \end{aligned} \tag{7}$$

The proof of Proposition 4 is given in the Supplementary Material A4. In addition, two- and three-level and two- and four-level RFLADs are also worthy to be discussed. As similar as the symmetrical case, the expression of the squared WD-value for the asymmetrical two- and three-level RFLAD is the same as (5), and the expression of WD for the two- and four-level RFLAD can be found in the Supplementary Material B4.

3 Lower bounds of level-augmented designs

Based on the expressions of WD for RELADs and RFLADs, one wants to search uniform design under some given parameters in practice. Thus it is necessary to derive the corresponding lower bound, which is the benchmark for searching uniform designs. In this section, we give the lower bounds of WD for symmetrical and asymmetrical level-augmented designs under some special parameters.

For any initial design $\boldsymbol{d}_0 \in \mathcal{U}(n; 2^{m_2}3^{m_1})$, we can add n_1 runs to obtain a symmetrical RELAD $D_1 \in \mathcal{L}_e(n+n_1; 3^m)$ or RFLAD $D_1' \in \mathcal{L}_f(n+n_1; 3^m)$. Since the expressions of the squared WD-value for those cases are the same, the corresponding lower bound of WD for the three-level RELAD and RFLAD is also the same and we obtain the lower bound as follows.

Theorem 1 *For any symmetrical range-extended level-augmented design in* $\mathcal{L}_e(n+n_1; 3^m)$ *or a symmetrical range-fixed level-augmented design in* $\mathcal{L}_f(n+n_1; 3^m)$*, the lower bound of the squared WD-value is*

$$\begin{aligned} LBW_1 = &-\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m \\ &+ \frac{1}{(n+n_1)^2}\Bigg[n(n-1)\left(\frac{3}{2}\right)^{\frac{(n-3)m_1}{3(n-1)}+\frac{(n-2)m_2}{2(n-1)}}\left(\frac{23}{18}\right)^{\frac{2nm_1}{3(n-1)}+\frac{nm_2}{2(n-1)}} \\ &+ n_1(n_1-1)\left(\frac{3}{2}\right)^{\frac{m_1(n_1-3)}{3(n_1-1)}+\frac{[3n_{11}(n_{11}-1)+(n_{12}-3)n_{12}]m_2}{3n_1(n_1-1)}}\left(\frac{23}{18}\right)^{\frac{2n_1m_1}{3(n_1-1)}+\frac{(3n_1^2-3n_{11}^2-n_{12}^2)m_2}{3n_1(n_1-1)}} \\ &+ 2nn_1\left(\frac{3}{2}\right)^{\frac{m_1}{3}+\frac{n_{12}m_2}{3n_1}}\left(\frac{23}{18}\right)^{\frac{2m_1}{3}+\frac{m_2(3n_1-n_{12})}{3n_1}}\Bigg]. \end{aligned} \tag{8}$$

This lower bound can be achieved if all its F_{ij}^{α} *distributions,* $i \neq j$*, are the same.*

The proof of Theorem 1 is given in the Appendix. Theorem 1 shows the lower bound for three-level level-augmented design. In this case, each F_{ij}^{α} distribution can be uniquely determined by the Hamming distance. The Hamming distance between the two rows is defined as the number of places where the two rows take different values. Thus the condition that all F_{ij}^{α} distributions are the same is equivalent to that this level-augmented design is a Hamming-equidistant design.

In addition, for any initial design $\boldsymbol{d}_0 \in \mathcal{U}(n; 2^m)$, we can add n_1 runs to obtain an asymmetrical RELAD $D_2 \in \mathcal{L}_e(n+n_1; 2^{m_1}3^{m_2})$ or RFLAD $D_2' \in \mathcal{L}_f(n+n_1; 2^{m_1}3^{m_2})$. Similarly, the expressions of the squared WD-value for them are the same. Then the same lower bound of WD for the asymmetrical two- and three-level RELAD and RFLAD is shown in the following theorem.

Theorem 2 *For any asymmetrical range-extended level-augmented design in* $\mathcal{L}_e(n+n_1; 2^{m_1}3^{m_2})$ *or asymmetrical range-fixed level-augmented design in* $\mathcal{L}_f(n+n_1; 2^{m_1}3^{m_2})$*, the lower bound of the squared WD-value is* $\max\{LBW_2, LBW_2'\}$*, where*

$$\begin{aligned} &LBW_2 \\ &= -\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)}\left(\frac{3}{2}\right)^m \\ &+ \frac{1}{(n+n_1)^2}\Bigg[n(n-1)\left(\frac{3}{2}\right)^{\frac{(n-2)m_1}{2(n-1)}+\frac{(n-2)m_2}{2(n-1)}}\left(\frac{5}{4}\right)^{\frac{nm_1}{2(n-1)}}\left(\frac{23}{18}\right)^{\frac{nm_2}{2(n-1)}} \end{aligned}$$

$$+ n_1(n_1-1)\left(\frac{3}{2}\right)^{\frac{m_1(n_1-2)}{2(n_1-1)}+\frac{[3n_{11}(n_{11}-1)+(n_{12}-3)n_{12}]m_2}{3n_1(n_1-1)}}\left(\frac{5}{4}\right)^{\frac{n_1m_1}{2(n_1-1)}}\left(\frac{23}{18}\right)^{\frac{(3n_1^2-3n_{11}^2-n_{12}^2)m_2}{3n_1(n_1-1)}}$$

$$+ 2nn_1\left(\frac{3}{2}\right)^{\frac{m_1}{2}+\frac{m_2n_{12}}{3n_1}}\left(\frac{5}{4}\right)^{\frac{m_1}{2}}\left(\frac{23}{18}\right)^{\frac{3n_1m_2-m_2n_{12}}{3n_1}}\Bigg], \tag{9}$$

LBW_2 can be achieved if all its F_{ij}^{α} distributions, $i \neq j$, are the same, and

$$LBW_2' = -\left(\frac{4}{3}\right)^m + \frac{1}{(n+n_1)^2}\left(\frac{5}{4}\right)^{m_1}\left(\frac{23}{18}\right)^{m_2}\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\binom{m_1}{i}\binom{m_2}{j}\left(\frac{1}{5}\right)^i\left(\frac{4}{23}\right)^j\theta_{ij}, \tag{10}$$

which was obtained by Chatterjee et al. (2005). Here, $\theta_{ij} = (n+n_1)g_{ij} + l_{ij}(g_{ij}+1)$, and $h_{ij} = 2^i3^j$, g_{ij} is the largest integer contained in $(n+n_1)/h_{ij}$, $l_{ij} = (n+n_1) - h_{ij}g_{ij}$, $0 \le i \le m_1$, $0 \le j \le m_2$.

The proof of Theorem 2 is given in the Appendix. Similar to the discussion in Fang et al. (2018), LBW_2 is often larger than LBW_2' and is easier to reach for saturated or supersaturated designs, while LBW_2' is often larger than LBW_2 and is more suitable for evaluating the uniformity of designs with large $n+n_1$ and small m. Moreover, even under the given parameters, the lower bounds of level-augmented designs in Theorems 1–2 may not be attainable. When the parameters meet the conditions in Theorem 1 or Theorem 2, the lower bounds of the corresponding level-augmented designs can be obtained directly.

4 Construction method

In this section, we give a method to construct three-level level-augmented designs. In the construction method, we utilized the results in Fang et al. (2005) that a three-level Hamming-equidistant design is also a uniform design under WD. The Hamming-equidistant design can be obtained only when some specific conditions for the parameters in Theorem 3 are satisfied. Assume that only one factor augments the number of levels, i.e. $m_2 = 1$. Let **1**, **2** and **3** denote the $n_0 \times 1$ vectors of 1s, 2s and 3s, respectively. The following steps can be used to construct the three-level RELAD D_1 and RFLAD D_1'.

Step 1. Given a three-level design $\boldsymbol{d} \in \mathcal{U}(n_0; 3^{m_0})$;
Step 2. Let $\phi^+(\boldsymbol{d}) = \boldsymbol{d} + 1(mod\ 3)$ and $\phi^-(\boldsymbol{d}) = \boldsymbol{d} + 2(mod\ 3)$;
Step 3. Let the initial mixed two- and three-level design have the form $\boldsymbol{d}_0 = \begin{pmatrix} \mathbf{1} & \boldsymbol{d} & \boldsymbol{d} & \boldsymbol{d} \\ \mathbf{2} & \boldsymbol{d} & \phi^+(\boldsymbol{d}) & \phi^-(\boldsymbol{d}) \end{pmatrix}$;
Step 4. Choose the additional portion $\boldsymbol{d}_1 = \big(\mathbf{3}\ \boldsymbol{d}\ \phi^-(\boldsymbol{d})\ \phi^+(\boldsymbol{d})\big)$ to obtain the symmetrical

RELAD $D_1 = \left(\boldsymbol{d}_0^T\ \boldsymbol{d}_1^T\right)^T$,

Step 5. Change the vector **2** in $\boldsymbol{d}_0$ into the vector **3** to obtain $\boldsymbol{d}_0'$ and choose the additional portion

$\boldsymbol{d}_1' = \left(\boldsymbol{2}\ \boldsymbol{d}\ \phi^-(\boldsymbol{d})\ \phi^+(\boldsymbol{d})\right)$ to obtain the symmetrical three-level RFLAD

$D_1' = \left(\boldsymbol{d}_0'^T\ \boldsymbol{d}_1'^T\right)^T$.

In the construction method, the number of the added runs n_0 is the minimal value that makes the resulting level-augmented design to be a U-type design. Hence it is reasonable to consider this case for saving cost. When the parameters satisfy some limitations which are shown in the following Theorem 3, three-level level-augmented designs D_1 and D_1' constructed by the above method have good uniformity.

Theorem 3 *(1) Consider the initial design* $\boldsymbol{d}_0 \in \mathcal{U}(2n_0; 2^1 3^{3m_0})$ *constructed by a uniform design* $\boldsymbol{d}$ *in* $\mathcal{U}(n_0; 3^{m_0})$*, as in Step 3 of the construction method, where* $m_0 = (n_0 - 1)/2$*. If* $2n_0 = 2 \cdot 3^{t-1}$, $t \geq 2$ *and* $3m_0 + 1 = m$*, then we add the additional portion* $\boldsymbol{d}_1$ *in* $\mathcal{D}(n_0; 3^1 3^{m_0})$*. The resulting three-level RELAD* D_1 *in* $\mathcal{L}_e(3n_0; 3^m)$ *can be a Hamming-equidistant design and hence the lower bound given in Theorem 1 is reachable.*

(2) If a three-level RELAD D_1 *in* $\mathcal{L}_e(3n_0; 3^{3(n_0-1)+1})$ *is constructed by* $\boldsymbol{d}$*, a uniform design in* $\mathcal{U}(n_0; 3^{n_0-1})$*, then the distribution of any two rows of* D_1 *is nearly the same, i.e. the difference of the Hamming distance of any two rows in* D_1 *is not more than one.*

The proof of Theorem 3 is given in the Appendix. Compared with D_1, the RFLAD D_1' constructed by Step 5 of the construction method just exchanges the position of the vectors **2** and **3**, hence it shall lead to the same results as in Theorem 3. In addition, in Step 3 of the construction method, it is necessary to verify whether $\boldsymbol{d}_0$ is a uniform or nearly uniform design or not. We first choose a uniform or nearly uniform design $\boldsymbol{d}$ in $\mathcal{U}(9; 3^{m_0})$ from the web http://web.stat.nankai.edu.cn/cms-ud/, where m_0 can take different values. Then we construct $\boldsymbol{d}_0$ with 18 runs and $3m_0+1$ columns using $\boldsymbol{d}$ and calculate the value of $WD^2(\boldsymbol{d}_0)$. According to the expression of the lower bound of WD for mixed-level designs in Zhou et al. (2008), we calculate the value of $LB(\boldsymbol{d}_0)$ and use the efficiency $D_{\text{eff}}(\boldsymbol{d}_0) = LB(\boldsymbol{d}_0)/WD^2(\boldsymbol{d}_0)$ to measure the uniformity of the constructed design. Then, we use the constructed $\boldsymbol{d}_0$ to construct D_1 with 27 runs and $3m_0 + 1$ columns by Step 4 of the construction method. By the formulas in Theorem 1, we calculate the lower bound of D_1 and measure the uniformity of D_1 by the efficiency $D_{\text{eff}}(D_1) = LB(D_1)/WD^2(D_1)$. The results are shown in Table 1 where $m_1 = 3m_0$. From the 4th and 6th columns of Table 1, most of the efficiencies are larger than 99% and both $\boldsymbol{d}_0$ and D_1 become more and more uniform as the number of m_1 increases. Thus it shows that both the constructed initial design $\boldsymbol{d}_0$ and the level-augmented design D_1 have good uniformity if $\boldsymbol{d}$ is a uniform or nearly uniform design. When $m_1 = 12$ and $m_2 = 1$, D_1 reaches the lower bound of WD and is a level-augmented uniform design. It meets the requirements of Theorem 3(1). When $m_1 = 6$ and $m_2 = 1$, the efficiency of D_1 is relatively small since Theorem 1 is more applicable to saturated or supersaturated designs.

Table 1 The values of $WD^2(\boldsymbol{d}_0)$, $D_{\text{eff}}(\boldsymbol{d}_0)$, $WD^2(D_1)$ and $D_{\text{eff}}(D_1)$

m		$WD^2(\boldsymbol{d}_0)$	$D_{\text{eff}}(\boldsymbol{d}_0)$	$WD^2(D_1)$	$D_{\text{eff}}(D_1)$
m_1	m_2				
6	1	1.0042	0.8836	0.7825	0.9053
12	1	11.6351	**1.0000**	9.3820	**1.0000**
18	1	137.9057	0.9754	107.0072	0.9787
24	1	1.5090×10^3	0.9977	1.1263×10^3	0.9982
30	1	1.6932×10^4	0.9911	1.2177×10^4	0.9923
33	1	5.6434×10^4	0.9942	3.9997×10^4	0.9951
36	1	1.8835×10^5	0.9972	1.3185×10^5	0.9979
45	1	7.1209×10^6	0.9973	4.8587×10^6	0.9978
51	1	8.0544×10^7	0.9981	5.4435×10^7	0.9985
57	1	9.1342×10^8	0.9988	6.1377×10^8	0.9991

Next, we give an example to illustrate the usefulness of the construction method, in which all the mentioned designs are shown in Example 1.

Example 2 Given $\boldsymbol{d} = (1\ 2\ 3)^T$, then $\phi^+(\boldsymbol{d}) = (2\ 3\ 1)^T$, $\phi^-(\boldsymbol{d}) = (3\ 1\ 2)^T$, and we construct the initial design $\boldsymbol{d}_0$ by Step 3. According to the expression of the squared WD-value for mixed two- and three-level design in Fang et al. (2018), $WD^2(\boldsymbol{d}_0) = 0.2571$ and its efficiency $D_{\text{eff}} = 99.92\%$, which shows that the initial design is a nearly uniform design. Based on Step 4, we get the additional portion $\boldsymbol{d}_1$ and $D_1 = (\boldsymbol{d}_0^T\ \ \boldsymbol{d}_1^T)^T$ is the symmetrical RELAD. By Step 5, we can also get $\boldsymbol{d}_0'$ and $\boldsymbol{d}_1'$. Then $D_1' = (\boldsymbol{d}_0'^T\ \ \boldsymbol{d}_1'^T)^T$ is the symmetrical RFLAD. According to Corollary 1, both $WD^2(D_1)$ and $WD^2(D_1')$ are equal to 0.1837, which reaches the lower bound in Theorem 1. Hence D_1 and D_1' are the symmetrical range-extended and range-fixed level-augmented uniform designs, respectively.

Given the initial design, if the conditions in Theorem 3 are not satisfied, the construction method may not work and the threshold accepting algorithm may be a suitable choice. It was widely used to search uniform designs, see Fang et al. (2003), Winker and Fang (1997). Its main idea is as follows. Given an initial design and the neighborhood, each iteration of the threshold accepting algorithm selects a new design randomly in the neighborhood of the current design. If the difference between the WD-value of the new design and that of the current design is less than or equal to a given threshold T_i in the ith iteration, then the current design is replaced by the new design. The threshold T_i is a nonnegative number and decreases to zero. The threshold accepting algorithm is a fast global searching method and can find the uniform or nearly uniform level-augmented designs. The following example is an asymmetrical case in which the designs are searched by the threshold accepting algorithm.

Example 3 Consider the following initial design $\boldsymbol{d}_0 \in \mathcal{U}(8; 2^4)$,

$$d_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

According to Fang et al. (2003), $WD^2(\boldsymbol{d}_0) = 0.4142$ and $\boldsymbol{d}_0$ reaches the lower bound of WD in $\mathcal{U}(8; 2^4)$. By the threshold accepting algorithm, we add the additional portion $\boldsymbol{d}_1 (\in \mathcal{D}(4; 3^4))$ as

$$\boldsymbol{d}_1 = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 3 & 1 & 2 & 2 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 2 & 1 \end{pmatrix},$$

and $D_2 = (\boldsymbol{d}_0^T \;\; \boldsymbol{d}_1^T)^T$ is the asymmetrical RELAD. In addition, we change the initial design to $\boldsymbol{d}_0'$, whose last three columns are the same as $\boldsymbol{d}_0$, and the first column is $(1, 1, 1, 1, 3, 3, 3, 3)^T$. We use the threshold accepting algorithm again to add the additional portion as

$$\boldsymbol{d}_1' = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix},$$

and $D_2' = (\boldsymbol{d}_0'^T \;\; \boldsymbol{d}_1'^T)^T$ is the asymmetrical RFLAD. According to Corollary 2, both $WD^2(D_2)$ and $WD^2(D_2')$ are equal to 0.3542, which reaches the lower bound given in Theorem 2. Thus D_2 and D_2' are the asymmetrical range-extended and range-fixed level-augmented uniform designs, respectively.

In addition, Table 2 also presents the summary of some uniform or nearly uniform level-augmented designs. The initial designs are all uniform designs or nearly uniform designs. The last three symmetrical level-augmented designs are constructed by the construction method and the rest of the designs are selected by using the threshold accepting algorithm. The last two columns show the squared WD-values and the efficiency of the level-augmented designs, respectively. It can be seen that the efficiencies of all the level-augmented designs are high and most of them are larger than 97%. All the level-augmented designs in Table 2 are listed in the Supplementary Material C.

5 Conclusion

In this paper, we discuss the level-augmented designs under WD. The level-augmented designs are applicable and useful when the number of the levels of some factors needs to be augmented in the follow-up stage. Based on the change of the experimental range,

Table 2 Summary of some uniform or nearly uniform level-augmented designs

Initial design	Level-augmented design	WD^2	D_{eff}
$U(4; 2^3)$	$D_2(6; 2^2 3^1)$, $D_2'(6; 2^2 3^1)$	0.1889	**1.0000**
	$D_2(6; 2^1 3^2)$, $D_2'(6; 2^1 3^2)$	0.1507	0.9768
$U(8; 2^4)$	$D_2(12; 2^2 3^2)$, $D_2'(12; 2^2 3^2)$	0.3051	0.9725
$U(8; 2^7)$	$D_2(12; 2^6 3^1)$, $D_2'(12; 2^6 3^1)$	1.6911	0.9846
	$D_2(12; 2^5 3^2)$, $D_2'(12; 2^5 3^2)$	1.5580	0.9718
$U(8; 2^{14})$	$D_2(12; 2^{13} 3^1)$, $D_2'(12; 2^{13} 3^1)$	34.0169	0.9919
	$D_2(12; 2^{12} 3^2)$, $D_2'(12; 2^{12} 3^2)$	32.7958	0.9900
$U(12; 2^{11})$	$D_2(18; 2^{10} 3^1)$, $D_2'(18; 2^{10} 3^1)$	9.4113	0.9823
	$D_2(18; 2^9 3^2)$, $D_2'(18; 2^9 3^2)$	8.8924	0.9798
$U(12; 2^{22})$	$D_2(18; 2^{21} 3^1)$, $D_2'(18; 2^{21} 3^1)$	690.3619	0.9970
	$D_2(18; 2^{20} 3^2)$,$D_2'(18; 2^{20} 3^2)$	675.9174	0.9951
$U(16; 2^8)$	$D_2(24; 2^7 3^1)$, $D_2'(24; 2^7 3^1)$	2.5874	0.9961
	$D_2(24; 2^6 3^2)$,$D_2'(24; 2^6 3^2)$	2.4086	0.9834
$U(16; 2^{15})$	$D_2(24; 2^{14} 3^1)$, $D_2'(24; 2^{14} 3^1)$	44.6417	0.9616
	$D_2(24; 2^{13} 3^2)$, $D_2'(24; 2^{13} 3^2)$	42.9649	0.9547
$U(16; 2^{30})$	$D_2(24; 2^{29} 3^1)$, $D_2'(24; 2^{29} 3^1)$	1.2858×10^4	0.9981
	$D_2(24; 2^{28} 3^2)$, $D_2'(24; 2^{28} 3^2)$	1.2704×10^4	0.9949
$U(18; 2^1 3^{12})$	$D_1(27; 3^{13})$, $D_1'(27; 3^{13})$	9.3820	**1.0000**
$U(18; 2^1 3^{18})$	$D_1(27; 3^{19})$, $D_1'(27; 3^{19})$	107.0072	0.9787
$U(54; 2^1 3^{40})$	$D_1(81; 3^{41})$, $D_1'(81; 3^{41})$	1.8098×10^5	**1.0000**

the level-augmented designs can be divided into range-extended and range-fixed level-augmented designs. From different types of initial designs, we define symmetrical and asymmetrical level-augmented designs for both range-extended and range-fixed cases.

The expressions of the squared WD-value for these level-augmented designs are derived. The lower bounds of WD for range-extended and range-fixed level-augmented designs are also obtained under some special parameters, which can be used as the benchmark for constructing level-augmented uniform designs. Moreover, we give a method to construct a special case of symmetrical level-augmented designs. In order to reduce the computational complexity, a further interesting question is to study a method to construct the general level-augmented designs. In addition, the level-augmented designs which contain both range-extended and range-fixed cases are also important, but more complex. It is beyond the scope of the current paper but worthy for further investigations.

Supplementary Materials

The proofs of all the propositions and the additional results are provided in the supplementary materials.

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Appendix

In order to prove Theorems 1 and 2, we give the following lemmas first. The proofs of Lemmas 1 and 2 are straightforward and are omitted.

Lemma 1 *For any symmetrical RELAD $D_1 \in \mathcal{L}_e(n+n_1; 3^m)$ or symmetrical RFLAD $D_1' \in \mathcal{L}_f(n+n_1; 3^m)$, we have*

$$(1) \sum_{i=1}^{n} \sum_{j\neq i=1}^{n} \varphi_{ij0} = \frac{m_1 n(n-3)}{3}, \ (2) \sum_{i=1}^{n} \sum_{j\neq i=1}^{n} \lambda_{ij0} = \frac{m_2 n(n-2)}{2},$$

$$(3) \sum_{i=n+1}^{n+n_1} \sum_{j\neq i=n+1}^{n+n_1} \varphi'_{ij0} = \frac{m_1 n_1(n_1-3)}{3},$$

$$(4) \sum_{i=n+1}^{n+n_1} \sum_{j\neq i=n+1}^{n+n_1} \lambda'_{ij0} = \left[n_{11}(n_{11}-1) + \frac{(n_{12}-3)n_{12}}{3}\right] m_2,$$

$$(5) \sum_{i=1}^{n} \sum_{j=n+1}^{n+n_1} \nu_{ij0} = \frac{m_1 n n_1}{3}, \ (6) \sum_{i=1}^{n} \sum_{j=n+1}^{n+n_1} \tau_{ij0} = \frac{m_2 n n_{12}}{3}.$$

Lemma 2 *For any asymmetrical RELAD $D_2 \in \mathcal{L}_e(n+n_1; 2^{m_1}3^{m_2})$ or asymmetrical RFLAD $D_2' \in \mathcal{L}_f(n+n_1; 2^{m_1}3^{m_2})$, we have*

$$(1) \sum_{i=1}^{n} \sum_{j\neq i=1}^{n} \varphi_{ij0} = \frac{m_1 n(n-2)}{2}, \ (2) \sum_{i=1}^{n} \sum_{j\neq i=1}^{n} \lambda_{ij0} = \frac{m_2 n(n-2)}{2},$$

$$(3) \sum_{i=n+1}^{n+n_1} \sum_{j\neq i=n+1}^{n+n_1} \varphi'_{ij0} = \frac{m_1 n_1(n_1-2)}{2},$$

$$(4) \sum_{i=n+1}^{n+n_1} \sum_{j\neq i=n+1}^{n+n_1} \lambda'_{ij0} = \left[n_{11}(n_{11}-1) + \frac{(n_{12}-3)n_{12}}{3}\right] m_2,$$

$$(5) \sum_{i=1}^{n} \sum_{j=n+1}^{n+n_1} \nu_{ij0} = \frac{m_1 n n_1}{2}, (6) \sum_{i=1}^{n} \sum_{j=n+1}^{n+n_1} \tau_{ij0} = \frac{m_2 n n_{12}}{3}.$$

Proof of Theorem 1. Based on Lemma 1, similar to the proof of Theorem 2.1 in Fang et al. (2005), we can obtain the result of Theorem 1 easily. A symmetrical RELAD $D_1 \in \mathcal{L}_e(n+n_1; 3^m)$ or a symmetrical RFLAD $D_1' \in \mathcal{L}_f(n+n_1; 3^m)$ is a uniform design under WD, if all its F_{ij}^{α} distributions, $i \neq j$, are the same. In this case, the WD-value of this design achieves the lower bound. □

Proof of Theorem 2. Based on Lemma 2, similar to the proof of Theorem 1, we can obtain the result of Theorem 2. □

Proof of Theorem 3. (1) Denote $(\,\mathbf{1}\;\; \boldsymbol{d}\;\; \boldsymbol{d}\;\; \boldsymbol{d}\,)$ as $\boldsymbol{d}_{01}$ and $\big(\,\mathbf{2}\;\; \boldsymbol{d}\;\; \phi^+(\boldsymbol{d})\;\; \phi^-(\boldsymbol{d})\,\big)$ as $\boldsymbol{d}_{02}$. According to Fang et al. (2005), since $\boldsymbol{d} = (\boldsymbol{d}_{ij})_{1\le i\le n_0, 1\le j\le m_0}$ is a three-level uniform design, it is also a Hamming-equidistant design and the coincidence number of any two rows in $\boldsymbol{d}$ is $\lambda_0 = m_0(n_0-3)/[3(n_0-1)]$. The coincidence number between two rows is defined as the number of places where two rows take the same value. The condition $n_0 = 3^{t-1}, t \ge 2$, ensures that λ_0 is an integer and the design $\boldsymbol{d}$ is available. Then, $\boldsymbol{d}_{01}$ is also a Hamming-equidistant design with coincidence number of any two rows being $3\lambda_0 + 1$. According to the definitions of $\phi^+(\boldsymbol{d})$ and $\phi^-(\boldsymbol{d})$ in Algorithm 1, they are obtained by the permutation of the levels of $\boldsymbol{d}$. Hence both the coincidence numbers of any two rows in $\boldsymbol{d}_{02}$ and in $\boldsymbol{d}_1$ are also $3\lambda_0 + 1$. For $i = 1, \ldots, n_0$, the ith rows of $\boldsymbol{d}$, $\phi^+(\boldsymbol{d})$ and $\phi^-(\boldsymbol{d})$ are different from each other and thus the coincidence number of any two among the ith rows in $\boldsymbol{d}_{01}$, $\boldsymbol{d}_{02}$ and $\boldsymbol{d}_1$, is m_0, the number of the columns of $\boldsymbol{d}$. The condition $m_0 = (n_0-1)/2$ implies that $m_0 = 3\lambda_0 + 1$. For $1 \le i \neq j \le n_0$, let $N_{ij}^c = \{k \mid \boldsymbol{d}_{ik} = \boldsymbol{d}_{jk}, k = 1, \ldots, m_0\}$ and $N_{ij}^h = \{1, \ldots, m_0\} - N_{ij}^c$. For any $k \in N_{ij}^c$ and $1 \le i \le n_0$, $\boldsymbol{d}_{ik}$, $\phi^+(\boldsymbol{d}_{ik})$ and $\phi^-(\boldsymbol{d}_{ik})$ become different from each other. However, for any $l \in N_{ij}^h$ and $1 \le i \neq j \le n_0$, if $\boldsymbol{d}_{jl}$ is one more than or two less than $\boldsymbol{d}_{il}$, $\phi^+(\boldsymbol{d}_{jl}) \neq \boldsymbol{d}_{il}$ and $\phi^-(\boldsymbol{d}_{jl}) = \boldsymbol{d}_{il}$; if $\boldsymbol{d}_{jl}$ is two more than or one less than $\boldsymbol{d}_{il}$, $\phi^+(\boldsymbol{d}_{jl}) = \boldsymbol{d}_{il}$ and $\phi^-(\boldsymbol{d}_{jl}) \neq \boldsymbol{d}_{il}$. Hence the coincidence number of the ith row in $\boldsymbol{d}_{01}$ and the jth row in $\boldsymbol{d}_{02}$ is also $m_0 (= 3\lambda_0 + 1)$ for $1 \le i \neq j \le n_0$. By similar arguments, we can obtain the same results for $\boldsymbol{d}_{01}$ and $\boldsymbol{d}_1$, as well as $\boldsymbol{d}_{02}$ and $\boldsymbol{d}_1$. Therefore, the resulting three-level level-augmented design D_1 is a Hamming-equidistant design and hence the lower bound given in Theorem 1 is reachable.

(2) Similar to the proof of (1), since $\boldsymbol{d}$ is a uniform design in $\mathcal{U}(n_0; 3^{n_0-1})$, it is also a Hamming-equidistant design and $\lambda_0 = (n_0-3)/3$. Since $\boldsymbol{d}$ is a U-type design, n_0 must be a multiple of 3 which ensures that λ_0 is an integer. The coincidence numbers of any two rows in $\boldsymbol{d}_{01}$, $\boldsymbol{d}_{02}$ and $\boldsymbol{d}_1$ are $n_0 - 2 (= 3\lambda_0 + 1)$. For $1 \le i \le n_0$, the coincidence number of any two among the ith rows in $\boldsymbol{d}_{01}$, $\boldsymbol{d}_{02}$ and $\boldsymbol{d}_1$, is $n_0 - 1$, the number of the columns of $\boldsymbol{d}$. For $1 \le i \neq j \le n_0$, the coincidence number of the ith row in $\boldsymbol{d}_{01}$ and the jth row in $\boldsymbol{d}_{02}$ is $n_0 - 1$, which is also true for $\boldsymbol{d}_{01}$ and $\boldsymbol{d}_1$, $\boldsymbol{d}_{02}$ and $\boldsymbol{d}_1$. Hence for D_1, the difference of the Hamming distances between its rows is not more than one. For both (1) and (2), the corresponding arguments for D_1' are similar to the case of D_1 and we omit it. □

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